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ON LINEAR DIFFERENTIAL GAMES

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1. Statement of the Problems

This paper treats a certain class of linear differential games, the state of which is specified by a state vector in an n-dimensional Euclidean space R^n. It is assumed that the state vector $z(t)$ in R^n can be expressed as

$$z(t)=z_1(t)+\int_0^t F(t,s)u(s)ds+\int_0^t W(t,s)v(s)ds, \tag{1}$$

where $u(t)$ and $v(t)$ are r-vectors called strategies or controls of the first and the second players, respectively, $F(t,s)$ and $W(t,s)$ are $n\times r$ matrix functions which are assumed to be continuous, and $z_1(t)$ is a known n-dimensional vector function. It is assumed that the strategies $u(t)$ and $v(t)$ of the first and second players are measurable and constrained as

$$u(t)\in U, \quad v(t)\in V, \tag{2}$$

where U and V are certain sets in an r-dimensional Euclidean space. Such strategies $u(t)$ and $v(t)$ which satisfy above mentioned conditions are called admissible strategies. In this paper, we treats the case where the sets U and V are unit cube, respectively, i.e.,

$$U=V=\Omega=\left\{u:\ |u_i|\leq 1 \quad (i=1,\cdots,r)\right\}. \tag{3}$$

We consider two kinds of differential game problems stated as follows:

Problem 1. Determine a saddle point for

$$J(u,v)= (z(T),z(T))+ \int_0^T\left\{(u(t),\, Cu(t)) - (v(t),\, Dv(t))\right\}dt\,, \qquad (4)$$

subject to the constraints (1) and (2), where $(\cdot,\cdot)$ denotes the inner product in some finite-dimensional Euclidean space, C and D are $r\times r$ positive semi-definite diagonal matrices with nonnegative constant elements, and T is a fixed time. $J(u, v)$ is called a payoff. A saddle point is defined as the pair $u^o(t)$, $v^o(t)$ satisfying the relation

$$J(u^o ,v)\leq J(u^o, v^o)\leq J(u, v^o) \qquad (5)$$

for arbitrary admissible strategies u and v. Namely, the first player is to select a strategy that minimizes the payoff and the second player is to select a strategy that maximizes the payoff. If (5) can be realized, u^o and v^o are called the optimal strategies and $J(u^o, v^o)$ is called the value of the game.

Problem 2. Let $T_{u,v}$ be a time corresponding to the strategies $u(t)\in U$, $v(t)\in V$ and satisfying $z(T_{u,v})= 0$. Determine a time T_{u^o,v^o} and a pair of admissible strategies $u^o(t)$, $v^o(t)$ such that

$$z(T_{u^o,v^o})= 0, \qquad T_{u^o,v^o}= \min_{u\in U}\max_{v\in V} T_{u,v}\,. \qquad (6)$$

The second equation of (6) is equivalent to

$$T_{u^o,v} \leq T_{u^o,v^o} \leq T_{u,v^o}\,.$$

Such differential game problems have been treated by several authors [1]-[7]. In this paper, we consider the problems as an infinite-dimensional non-linear programming problem [8]. By using the generalized Kuhn-Tucker theorem [9] in nonlinear programming, we derive a system of transcendental equations, the solution of which directly yields the optimal strategies.

We can mention two kinds of games describable in the form (1). One is a pursuit-evasion game governed by

$$\left.\begin{aligned} dx_p/dt &= A_p(t)x_p + B_p(t)u(t) + h_p(t), \quad x_p(0)= x_{po}, \\ dx_e/dt &= A_e(t)x_e + B_e(t)v(t) + h_e(t), \quad x_e(0)= x_{eo}, \end{aligned}\right\} \tag{7}$$

where x_p is an n-vector representing the state of the pursuer, $u(t)\in U$ is an r-vector representing the control of the pursuer, $A_p(t)$, $B_p(t)$, and $h_p(t)$ are $n\times n$, $n\times r$, and $n\times 1$ known matrices, respectively, continuous in t, and identical statements apply to the evader and x_e, $v(t)\in V$, $A_e(t)$, $B_e(t)$, and $h_e(t)$. x_{po} and x_{eo} are initial values. The state of the game is defined by

$$z(t)= Q\left\{x_p(t)- x_e(t)\right\}, \tag{8}$$

where Q is an $n\times n$ positive semidefinite constant matrix. Then, the $z_1(t)$, $F(t,s)$, and $W(t,s)$ in (1) are given by

$$\left.\begin{aligned} z_1(t) &= Q\left\{ X_p(t)\left(x_{po}+ \int_0^t X_p^{-1}(s)h_p(s)ds\right) - X_e(t)\left(x_{eo}+\int_0^t X_e^{-1}(s)h_e(s)ds\right)\right\}, \\ F(t, s) &= QX_p(t)X_p^{-1}(s)B_p(s), \\ W(t, s) &= -QX_e(t)X_e^{-1}(s)B_e(s), \end{aligned}\right\} \tag{9}$$

where $X_p(t)$ and $X_e(t)$ are $n\times n$ matrix functions satisfying

$$\left.\begin{aligned} dX_p(t)/dt &= A_p(t)X_p(t), \quad X_p(0)= I \text{ (the identity)}, \\ dX_e(t)/dt &= A_e(t)X_e(t), \quad X_e(0)= I. \end{aligned}\right\}$$

The other game describable in the form (1) is a control system subject to unpredictable disturbances. The state of the control system is governed by

$$dz/dt = A(t)z+ B_1(t)u(t)+ B_2(t)v(t)+ h(t), \quad z(0)= z_o, \tag{10}$$

where $z(t)$ is an n-vector describing the state of the system, $u(t)\in U$ is an r-vector representing the control, $v(t)$ is an r-vector representing unpredictable disturbance functions, $A(t)$, $B_1(t)$, $B_2(t)$, and $h(t)$ are $n\times n$, $n\times r$, $n\times r$, and $n\times 1$ known matrices, respectively, which are continuous in t. The known information concerning $v(t)$ is only the fact that $v(t)\in V$. In this case, the $z_1(t)$, $F(t, s)$, and $W(t, s)$ in (1) are given by

$$\left.\begin{aligned} z_1(t) &= X(t)\,(z_0 + \int_0^t X^{-1}(s)h(s)ds\,), \\ F(t, s) &= X(t)\,X^{-1}(s)B_1(s), \\ W(t, s) &= X(t)\,X^{-1}(s)B_2(s), \end{aligned}\right\} \tag{11}$$

where z_0 is the initial value of $z(t)$, and $X(t)$ is an $n\times n$ matrix function satisfying

$$dX(t)/dt = A(t)X(t), \quad X(0) = I.$$

2. Fundamental Theorem and its Application to the Problem

Since the solution of the Problem 2 is obtained from the solution of the Problem 1, we consider the Problem 1 first. Let H be a real Hilbert space of r-dimensional functions square integrable over $[0, T]$. Then, the strategies $u(t)$ and $v(t)$ $(0\le t\le T)$ can be taken in H. Define the inner product of two vectors u^1 and u^2 in the Hilbert space H by

$$[u^1, u^2] = \int_0^T u^{1*}(t)u^2(t)dt = \int_0^T (u^1(t), u^2(t))dt,$$

where * denotes the transpose of a vector or a matrix. We define linear operators P and R, respectively, by

$$\left.\begin{aligned} Pu &= \int_0^T F(T, s)u(s)ds, \\ Rv &= \int_0^T W(T,s)v(s)ds, \end{aligned}\right\} \tag{12}$$

which map the Hilbert space H into the n-dimensional Euclidean space R^n. Then, from (1) it follows that

$$z_T = z_1 + Pu + Rv, \tag{13}$$

where $z_T = z(T)$ and $z_1 = z_1(T)$. The payoff (4) is rewritten as

$$J(u, v) = (z_T, z_T) + [u, Cu] - [v, Dv]. \tag{14}$$

By using (13), it follows that

$$\begin{aligned}(z_T, z_T) &= (Pu + Rv + z_1, Pu + Rv + z_1) \\ &= (Pu, Pu) + 2(Pu, Rv + z_1) + (Rv + z_1, Rv + z_1).\end{aligned} \tag{15}$$

Let P^* now be the adjoint operator of P, then P^* maps R^n into H and satisfies the relation

$$(x, Pu) = [P^*x, u],$$

where $x \in R^n$ and $u \in H$. From (12), it can be easily seen that [10]

$$P^*x = F^*(T, t)x, \tag{16}$$

where F^* denotes the transposed matrix of F. Then, (15) can be rewritten as

$$(z_T, z_T) = [u, P^*Pu] + 2[u, P^*(Rv + z_1)] + (Rv + z_1, Rv + z_1).$$

Therefore, the payoff (14) can be written as

$$\begin{aligned}J(u, v) &= [u, (P^*P + C)u] + 2[u, P^*(Rv + z_1)] + (Rv + z_1, Rv + z_1) - [v, Dv] \\ &= [v, (R^*R - D)v] + 2[v, R^*(Pu + z_1)] + (Pu + z_1, Pu + z_1) + [u, Cu].\end{aligned} \tag{17}$$

Since $[u, (P^*P + C)u] = (Pu, Pu) + [u, Cu] \geq 0$, it can be seen that $J(u, v)$ is convex with respect to $u \in H$. But the convexity or concavity of $J(u, v)$ with respect to $v \in H$ cannot be asserted. Define a mapping g, which maps H_1 ($H_1 \subset H$) into H, by

$$g(u) = \begin{pmatrix} 1 - u_1^2(t) \\ 1 - u_2^2(t) \\ \vdots \\ 1 - u_r^2(t) \end{pmatrix}. \tag{18}$$

Then, the constraints $u(t),\ v(t) \in \Omega$ can be expressed as

$$g(u) \geq 0, \quad g(v) \geq 0 . \tag{19}$$

Define a closed bounded convex subset X of H by

$$X = \{ u \in H : g(u) \geq 0 \} .$$

Since $J(u, v)$ is continuous and convex with respect to u on X, from [11, Theorem 2.1] there exists an element u^o in X such that

$$\inf_{u \in X} J(u, v) = J(u^o, v) .$$

Concerning an element v^o in X which maximizes $J(u, v)$, the existence cannot be asserted. Therefore, we assume the existence of v^o. In the following, necessary conditions for the optimal strategies will be derived.

First, we show a theorem which corresponds to a special case of the Theorem V.3.3.2 in [9] and is available for our problem.

Theorem 1. : Let f be a real-valued differentiable functional on the Hilbert space H. Let $x^o \in X$ maximize $f(x)$ subject to the constraint $g(x) \geq 0$. Then, there exists a $\lambda^o \in H$ such that

$$\lambda^o \geq 0 , \tag{20}$$

and that the Lagrangian expression

$$\Phi(x, \lambda) = f(x) + [\lambda , g(x)] \tag{21}$$

satisfies the following relations:

$$\delta_x \Phi((x^o, \lambda^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{22}$$

$$[\lambda^o, g(x^o)] = 0 , \tag{23}$$

where $\delta_x \Phi((x^o, \lambda^o); \xi)$ represents the partial Fréchet differential of Φ with respect to x at (x^o, λ^o) with increment ξ, which is defined by

$$\delta_x \bar{\Phi}((x^o, \lambda^o); \xi) = \lim_{\varepsilon \to 0} \frac{\Phi(x^o + \varepsilon\xi, \lambda^o) - \Phi(x^o, \lambda^o)}{\varepsilon} . \tag{24}$$

Proof. Since the function g defined by (18) is regular [9] at every point of X, it follows from [9, Theorem V.3.3.1] that the relation

$$\left.\begin{array}{l} \delta g(x^o; \xi) + g(x^o) \geq 0 \\ \text{implies} \\ -\delta f(x^o; \xi) \geq 0 . \end{array}\right\} \tag{25}$$

Define a linear continuous transformation $A(\xi)$ on H into H by

$$A(\xi) = \delta g(x^o; \xi) , \tag{26}$$

and a linear functional $\phi(\xi)$ on H by

$$\phi(\xi) = -\delta f(x^o; \xi) . \tag{27}$$

Since A is bounded, it follows that a subset Y_A in H defined by

$$Y_A = \left\{ A^*(\lambda) \in H : \lambda \in H, \ \lambda \geq 0 \right\} \tag{28}$$

is regularly convex [9] , where A^* is the adjoint transformation of A. Further,

$$A(\xi) + g(x^o) \geq 0 \quad \text{implies} \quad \phi(\xi) \geq 0 .$$

Therefore, from [9, Corollary IV. 3], there exists a $\lambda^o \geq 0$ ($\lambda^o \in H$) such that

$$[\lambda^o, A(\xi)] = \phi(\xi) \quad \text{for all} \quad \xi \in H .$$

Furthermore,

$$[\lambda^o, g(x^o)] = 0 .$$

Thus, the Theorem 1 is proved.

Now, we apply Theorem 1 to our problem. Let $u^o \in X$ minimize $J(u, v)$ subject to the constraint $g(u) \geq 0$, then there exists a $\lambda^o \in H$ such that

$$\lambda^o \geq 0 , \tag{29}$$

and that the Lagrangian expression

$$K(u, v, \lambda) = J(u, v) - [\lambda, g(u)] \tag{30}$$

satisfies the following relations:

$$\delta_u K((u^o, v, \lambda^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{31}$$

$$[\lambda^o, g(u^o)] = 0 . \tag{32}$$

Analogously, let $v^o \in X$ maximize $J(u, v)$ subject to the constraint $g(v) \geq 0$, then there exists a $\mu^o \in H$ such that

$$\mu^o \geq 0 , \tag{33}$$

and that the Lagrangian expression

$$L(u, v, \mu) = J(u, v) + [\mu, g(v)] \tag{34}$$

satisfies the following relations:

$$\delta_v L((u, v^o, \mu^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{35}$$

$$[\mu^o, g(v^o)] = 0 . \tag{36}$$

Henceforth, u^o, v^o, λ^o, and μ^o which satisfy these relations are simply written as u, v, λ, and μ, since no confusion may occur. In view of (17) and (24), the partial Fréchet differential of $K(u, v, \lambda)$ with respect to u at (u, v, λ) with increment ξ can be evaluated as

$$\delta_u K((u, v, \lambda); \xi) = 2\left[(P^*P + C)u, \xi\right] + 2\left[P^*(Rv + z_1), \xi\right] - \left[\frac{\partial g(u)}{\partial u}\lambda, \xi\right] ,$$

where $\partial g(u)/\partial u$ denotes an rxr matrix defined by

$$\frac{\partial g(u)}{\partial u} = \left[\frac{\partial g_i(u)}{\partial u_j}\right] = -2 \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots, & u_r \end{bmatrix} .$$

Hence, from (31) it follows that

$$(P^*P+C)u+P^*Rv+P^*z_1-\frac{1}{2}\frac{\partial g(u)}{\partial u}\lambda=0\,. \tag{37}$$

Define a new vector x in H by

$$x=Cu-\frac{1}{2}\frac{\partial g(u)}{\partial u}\lambda=\begin{bmatrix} c_1u_1+\lambda_1u_1 \\ \vdots \\ c_ru_r+\lambda_ru_r \end{bmatrix}, \tag{38}$$

where c_i $(i=1,\cdots,r)$ are the elements of the diagonal matrix C and non-negative. Then, (37) is rewritten as

$$P^*(Pu+Rv+z_1)+x=0\,. \tag{39}$$

In the same way, since

$$\delta_v L((u,v,\mu);\xi)=2\left[(R^*R-D)v,\xi\right]+2\left[R^*(Pu+z_1),\xi\right]+\left[\frac{\partial g(v)}{\partial v}\mu,\xi\right],$$

it follows from (35) that

$$(R^*R-D)v+R^*Pu+R^*z_1+\frac{1}{2}\frac{\partial g(v)}{\partial v}\mu=0\,. \tag{40}$$

Define a new vector y in H by

$$y=Dv-\frac{1}{2}\frac{\partial g(v)}{\partial v}\mu=\begin{bmatrix} d_1v_1+\mu_1v_1 \\ \vdots \\ d_rv_r+\mu_rv_r \end{bmatrix}, \tag{41}$$

where d_i $(i=1,\cdots,r)$ are the elements of the diagonal matrix D and nonnegative. Then, (40) is rewritten as

$$R^*(Rv+Pu+z_1)-y=0\,. \tag{42}$$

Since $\lambda(t)\geq 0$ and $g(u(t))\geq 0$ on the interval $[0,T]$, it follows from (32) that the equation

$$(\lambda(t),\ g(u(t)) = \lambda^*(t) g(u(t)) = 0$$

holds for almost all $t \in [0, T]$. Therefore,

$$\left.\begin{array}{lll} \lambda_i(t) = 0 & \text{if} & -1 < u_i(t) < 1, \\ \lambda_i(t) \geq 0 & \text{if} & |u_i(t)| = 1. \end{array}\right\}$$

Thus, the relation between $u_i(t)$ and $\lambda_i(t)$ can be shown as Fig. 1 (a). The relation between $u_i(t)$ and $\lambda_i(t)u_i(t)$ and then the relation between $u_i(t)$ and $c_i(t)u_i(t) + \lambda_i(t)u_i(t)$ can also be obtained successively from Fig. 1 (a) as shown in Fig. 1 (b) and (c), respectively. Hence, the relation between $x \in H$ and $u \in X$, which is defined by (38), can be expressed as

$$u(t) = \varphi(x(t)), \quad \text{or} \quad u_i(t) = \varphi_i(x_i(t)) \quad (i = 1, \cdots, r), \tag{43}$$

where the nonlinear function φ_i is shown in Fig. 2 (a), which can be obtained from Fig. 1 (c) directly.

Analogously, the relation between $y \in H$ and $v \in X$, which is defined by (41), can be expressed as

$$v(t) = \psi(y(t)), \quad \text{or} \quad u_i(t) = \psi_i(y_i(t)) \quad (i = 1, \cdots, r), \tag{44}$$

where the nonlinear function ψ_i is shown in Fig. 2 (b). If we use a notation such that

$$\left.\begin{array}{lll} \text{sat}\,\alpha = \alpha & \text{if} & |\alpha| \leq 1, \\ \text{sat}\,\alpha = \text{sgn}\,\alpha & \text{if} & |\alpha| \geq 1, \end{array}\right\}$$

then,

$$\varphi(x) = \begin{bmatrix} \text{sat}(x_1/\ c_1) \\ \vdots \\ \text{sat}(x_r/\ c_r) \end{bmatrix}, \qquad \psi(y) = \begin{bmatrix} \text{sat}(y_1/\ d_1) \\ \vdots \\ \text{sat}(y_r/\ d_r) \end{bmatrix}.$$

Substituting (43) and (44) into (39) and (42) yields

$$\left.\begin{aligned} P^*(P\varphi(x)+R\psi(y)+z_1)+x=0, \\ R^*(P\varphi(x)+R\psi(y)+z_1)-y=0. \end{aligned}\right\} \tag{45}$$

Equation (45) is a system of nonlinear integral equations from which $x \in H$ and $y \in H$ can be determined. This system of nonlinear integral equations can be reduced to a system of transcendental equations as follows.

By setting

$$P\varphi(x)+R\psi(y)+z_1=\alpha , \tag{46}$$

we obtain

$$x=-P^*\alpha \quad , \qquad y=R^*\alpha \quad , \tag{47}$$

where α is a vector in R^n. In view of (13), it is clear that the α defined by (46) represents $z_T=z(T)$. Substituting (47) into (46) yields a transcendental equation:

$$-P\varphi(P^*\alpha)+R\psi(R^*\alpha)+z_1=\alpha . \tag{48}$$

Let $f_i(T, t)$ and $w_i(T, t)$ $(i=1, \cdots, r)$ be n-dimensional column vectors of the matrices $F(T, t)$ and $W(T, t)$, respectively, i.e.,

$$\left.\begin{aligned} F(T, t) &= \left[f_1(T, t) \quad f_2(T, t) \quad \cdots \quad f_r(T, t)\right] , \\ W(T, t) &= \left[w_1(T, t) \quad w_2(T, t) \quad \cdots \quad w_r(T, t)\right] . \end{aligned}\right\}$$

Then, since

$$P^*\alpha = F^*(T, t)\alpha = \begin{bmatrix} f_1^*(T, t)\alpha \\ \vdots \\ f_r^*(T, t)\alpha \end{bmatrix}, \quad R^*\alpha = W^*(T, t)\alpha = \begin{bmatrix} w_1^*(T, t)\alpha \\ \vdots \\ w_r^*(T, t)\alpha \end{bmatrix},$$

(48) can be rewritten as

$$\alpha = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\psi_i(w_i^*(T,t)\alpha) - f_i(T,t)\varphi_i(f_i^*(T,t)\alpha) \Big] dt + z_1 . \tag{49}$$

3. Solution of the Transcendental Equation

Let us define a mapping A by

$$A\alpha = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\psi_i(w_i^*(T,t)\alpha) - f_i(T,t)\varphi_i(f_i^*(T,t)\alpha) \Big] dt + z_1 , \tag{50}$$

which maps R^n into R^n. Let $(A\alpha)_j$, f_{ij}, w_{ij}, and z_{1j} denote the jth component of n-vectors $A\alpha$, f_i, w_i, and z_1, respectively. Then,

$$(A\alpha)_j = \sum_{i=1}^{r} \int_0^T \Big[w_{ij}(T,t)\psi_i(w_i^*(T,t)\alpha) - f_{ij}(T,t)\varphi_i(f_i^*(T,t)\alpha) \Big] dt + z_{1j} .$$

Since $|\varphi_i| \le 1$, $|\psi_i| \le 1$ $(i = 1, \cdots, r)$, it follows that

$$\left.\begin{aligned} |(A\alpha)_j| \le \sum_{i=1}^{r} \int_0^T \Big[|w_{ij}(T,t)| + |f_{ij}(T,t)| \Big] dt + |z_{1j}| \\ (j = 1, \cdots, n) . \end{aligned}\right\} \tag{51}$$

Since the functions f_{ij}, w_{ij} $(i = 1, \cdots, r;\ j = 1, \cdots, n)$ are assumed to be continuous on the closed interval $0 \le t \le T$, (51) shows that the mapping A maps a closed bounded convex subset of R^n into itself. Furthermore, the mapping is continuous. Therefore, by Brouwer's fixed-point theorem [12], we can conclude that there exists a point α such that $A\alpha = \alpha$. Namely, Theorem 2. Let the vector functions f_i and w_i $(i=1,\cdots,r)$ be continuous on the closed interval $[0, T]$. Then, there exists a solution of (49).

The solution of (49) may be computed by the method of successive approximations:

$$\alpha_k = A\alpha_{k-1} \qquad (k=1,2,\cdots). \tag{52}$$

As to the convergence of the successive approximations (52), we can propose:

Theorem 3. Let the nonnegative constants c_i, d_i $(i=1,\cdots,r)$ be all positive. Further, let us assume that

$$\sum_{i=1}^{r}\int_0^T\left[\frac{1}{c}\|f_i(T,t)\|^2+\frac{1}{d}\|w_i(T,t)\|^2\right]dt<1, \tag{53}$$

where $c=\min(c_1, c_2,\cdots, c_r)$, $d=\min(d_1, d_2,\cdots, d_r)$, and $\|\ \|$ denotes the Euclidean norm in R^n. Then, the successive approximations (52), starting with an arbitrary α_0, converge to a unique solution of (49).

Proof. Let α and β be arbitrary points in R^n. By using the Schwarz inequality and the relations:

$$\left.\begin{aligned}
&|\varphi_i(f_i^*(T,t)\alpha)-\varphi_i(f_i^*(T,t)\beta)|\le\frac{1}{c}|f_i^*(T,t)(\alpha-\beta)|\\
&\qquad\qquad\le\frac{1}{c}\|f_i(T,t)\|\,\|\alpha-\beta\|,\\
&|\psi_i(w_i^*(T,t)\alpha)-\psi_i(w_i^*(T,t)\beta)|\le\frac{1}{d}\|w_i(T,t)\|\,\|\alpha-\beta\|,\\
&\qquad (i=1,\cdots,r)
\end{aligned}\right\} \tag{54}$$

it follows that

$$|(A\alpha)_j-(A\beta)_j|\le\|\alpha-\beta\|\sum_{i=1}^{r}\int_0^T\left[\frac{1}{c}|f_{ij}(T,t)|\,\|f_i(T,t)\|+\frac{1}{d}|w_{ij}(T,t)|\,\|w_i(T,t)\|\right]dt$$

$$\leq \|\alpha-\beta\| \left[\frac{1}{c} \left\{ \int_0^T \sum_{i=1}^{r} f_{ij}^2 (T,t)\,dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^{r} \| f_i(T,t)\|^2 dt \right\}^{\frac{1}{2}} \right.$$

$$\left. + \frac{1}{d} \left\{ \int_0^T \sum_{i=1}^{r} w_{ij}^2 (T,t)\,dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^{r} \| w_i(T,t)\|^2 dt \right\}^{\frac{1}{2}} .\right.$$

Hence,

$$\|A\alpha - A\beta\| \leq \left[\frac{1}{c} \int_0^T \sum_{i=1}^{r} \| f_i(T,t)\|^2 dt \right.$$

$$\left. + \frac{1}{d} \int_0^T \sum_{i=1}^{r} \| w_i(T,t)\|^2 dt \right] \|\alpha-\beta\| . \tag{55}$$

Inequality (55) shows that the mapping A defined by (50) is a contraction mapping under the condition (53) [13]. Thus, the theorem is proved.

4. Solution to the Problem 2

If $C=D=0$ in (4), the payoff becomes

$$\hat{J}(u, v) = (z(T), z(T)) . \tag{56}$$

In this case, equation (49) which determines the vector α becomes

$$\alpha = \sum_{i=1}^{r} \int_0^T \left[w_i(T,t)\ \mathrm{sgn}\left(w_i^*(T,t)\alpha \right) \right.$$

$$\left. - f_i(T,t)\,\mathrm{sgn}\left(f_i^*(T,t)\alpha \right) \right] dt + z_1 . \tag{57}$$

Let us introduce a real number ε and a vector β in R^n such that

$$\alpha = \varepsilon\beta , \quad \varepsilon > 0 , \quad \|\beta\| = 1 .$$

Then, (57) can be rewritten as

$$\varepsilon \beta = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \mathrm{sgn}\big(w_i^*(T,t)\beta\big) - f_i(T,t)\, \mathrm{sgn}\big(f_i^*(T,t)\beta\big) \Big] dt + z_1 . \tag{58}$$

As mentioned before, the vector α defined by (46) represents $z_T = z(T)$. Hence, if there exists a time T such that the value $\hat{J}(u^o, v^o)$ of the game vanishes, then there exists a solution of the Problem 2. Letting $\varepsilon \to 0$ in (58) yields a equation

$$\sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \mathrm{sgn}\big(w_i^*(T,t)\beta\big) - f_i(T,t)\, \mathrm{sgn}\big(f_i^*(T,t)\beta\big) \Big] dt + z_1 = 0 . \tag{59}$$

The time T and the n-vector β $(\|\beta\| = 1)$ which satisfy (59) give a solution to the Problem 2, i.e.,

$$T_{u^o, v^o} = T . \tag{60}$$

Furthermore, from (43), (44), and (47) it follows that

$$u^o(t) = -\begin{bmatrix} \mathrm{sgn}\,(f_1^*(T,t)\beta) \\ \vdots \\ \mathrm{sgn}\,(f_r^*(T,t)\beta) \end{bmatrix}, \quad v^o(t) = \begin{bmatrix} \mathrm{sgn}\,(w_1^*(T,t)\beta) \\ \vdots \\ \mathrm{sgn}\,(w_r^*(T,t)\beta) \end{bmatrix}. \tag{61}$$

References

(1) L. D. Berkovitz and W. H. Fleming, On differential games with integral payoff, in M. Dresher, A. W. Tucker, and P. Wolfe (eds.), Contributions to the Theory of Games III, Princeton U. P., 1957, pp. 413-435.

(2) L. D. Berkovitz, A variational approach to differential games, in M. Dresher, L. S. Shapley, and A. W. Tucker (Eds.), Advances in Game Theory, Princeton U. P., 1964, pp. 127-174.

(3) R. Isaacs, Differential Games, John Wiley, 1965.

(4) D. L. Kelendzheridze, A pursuit problem, in Pontryagin et al., The Mathematical Theory of Optimal Processes, Interscience Publishers, 1962, pp. 226-237.

(5) D. L. Kelendzheridze, On a problem of optimum tracking, Avtomat. i Telemek., Vol. 23, 1962, pp. 1008-1013.

(6) L. S. Pontryagin, On some differential games, Dokl. Akad. Nauk SSSR, Vol. 156, 1964, pp. 738-741.

(7) Y. C. Ho, A. E. Bryson, and S. Baron, Differential games and optimal pursuit-evasion strategies, IEEE Trans. on Automatic Control, Vol. AC-10, No. 4, 1965, pp. 385-389.

(8) Y. Sakawa, On a solution of an optimization problem in linear systems with quadratic performance index, J. SIAM Control, Vol. 4, No. 2, 1966, pp. 382-395.

(9) L. Hurwicz, Programming in linear spaces, in K. J. Arrow, L. Hurwicz, and H. Uzawa, Studies in Linear and Non-linear Programming, Stanford U. P., 1958, pp. 38-102.

(10) A. V. Balakrishnan, An operator theoretic formulation of a class of control

problems and a steepest descent method of solution, J. SIAM Control, Vol. 1, No. 2, 1963, pp. 109-127.

(11) A. V. Balakrishnan, Optimal control problems in Banach spaces, J. SIAM Control, Vol. 3, No. 1, 1965, pp. 152-180.

(12) T. L. Saaty and J. Bram, Nonlinear Mathematics, McGraw-Hill, 1964.

(13) A. N. Kolmogorov and S. V. Formin, Elements of the Theory of Functions and Functional Analysis, Graylock Press, 1957.

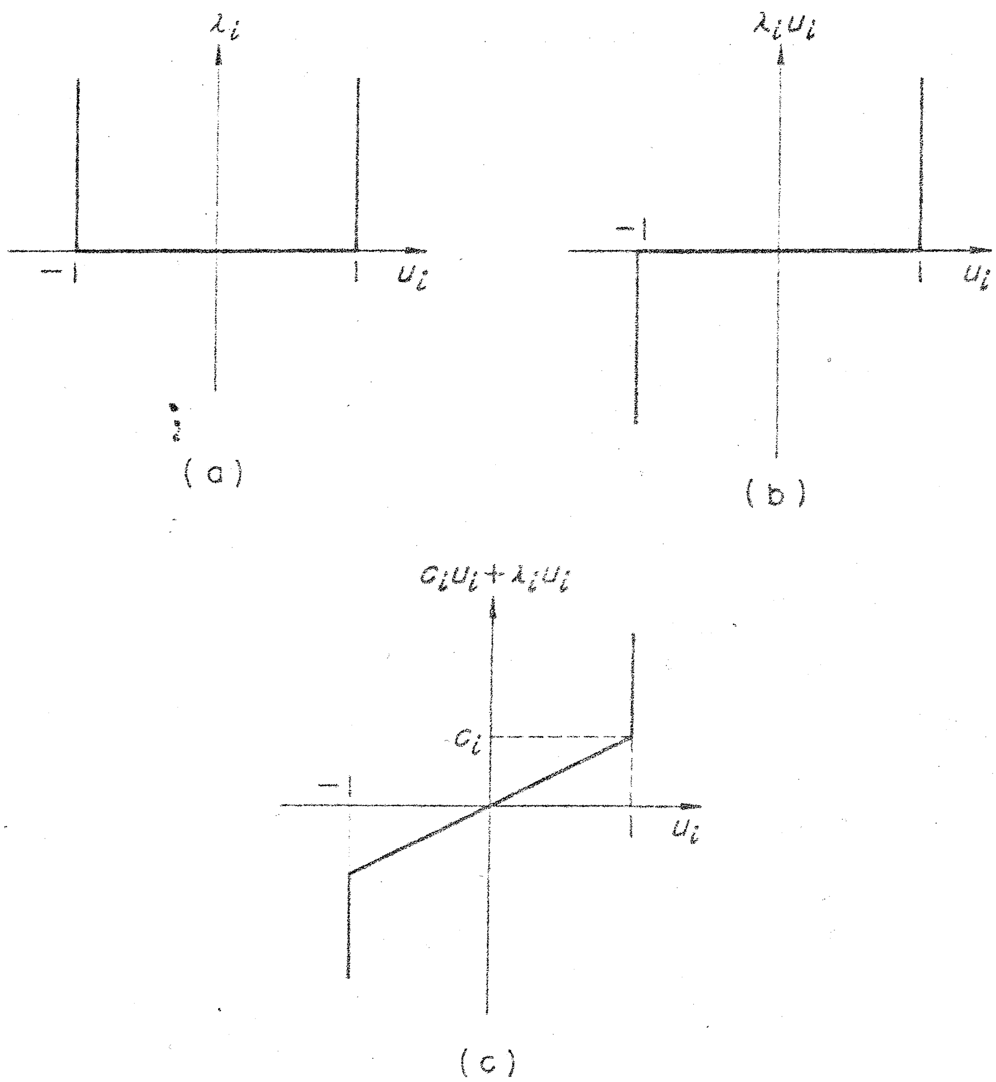

Fig. 1 Relations between the variables.

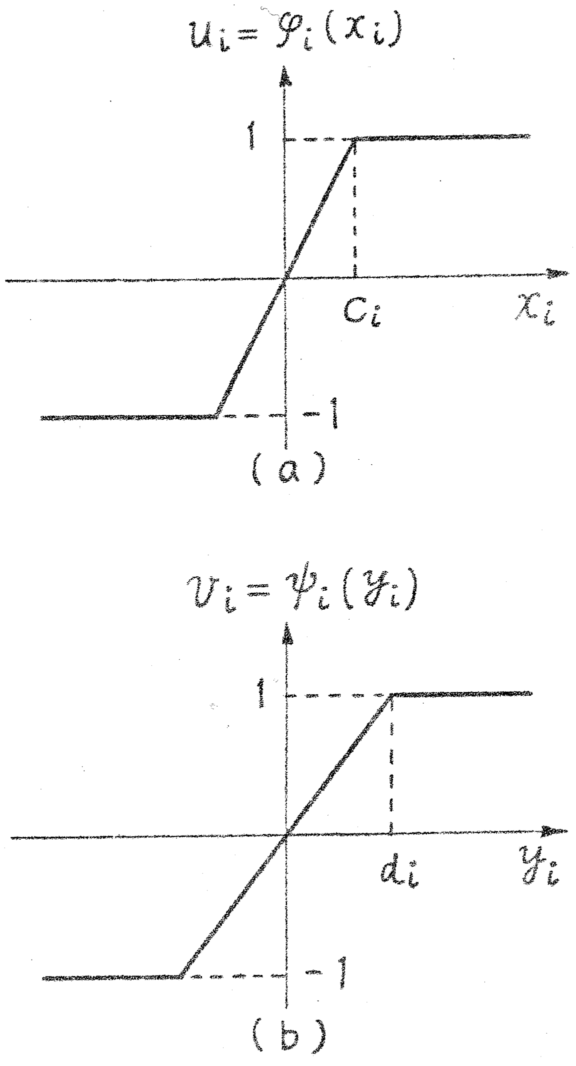

Fig. 2 Nonlinear characteristics.